Open book decomposition -lecture notes-

- Introduction, examples, existence of OBD ~~[illegible]~~ and contact structure on 3-mflds

(1.) Def: Open book decomposition of a smooth orientable mfd M is a pair (K,θ) with

$K \subset M$ codim 2 submfd with trivial normal bundle $N \overset{f}{\cong} \mathbb{D}^2 \times K$

$\theta : M_{\setminus K} \to S^1$ fiber bundle s.t.

in N wrt f, θ is just $(z,x) \to \frac{z}{\|z\|}$

K is called binding

$\theta^{-1}(t)$ pages $\forall t \in S^1$

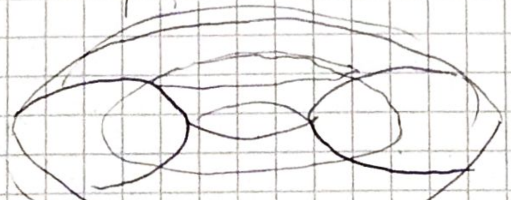
Examples: 1) $(\mathbb{R}^2, *, \theta)$

$\theta(z) = \frac{z}{\|z\|}$

2) 1) induces OBD $(\mathbb{R}^n, \{*\} \times \mathbb{R}^{n-2}, \theta)$

3) in particular, for $n=3$ after one-pt-compactification we get OBD on S^3 over unknot

In this case nbhd of the binding and pages decompose S^3 into two solid tori:

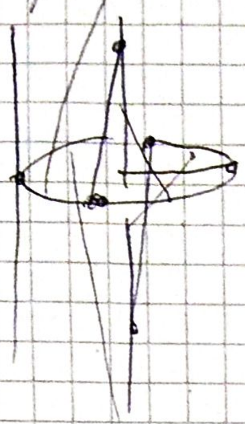


4) $S^2 \times S^1$

$\{N, S\} \subset S^2$, $S^2_{\setminus \{N,S\}} \cong S^1 \times (0,1)$

π_1: $S^1 \times (0,1) \times S^1 \to S^1$ gives OBD

5) Hopf link OBD



~~page is genus one~~ ~~su~~
page is annulus
monodromy is Dehn twist
(see abstract OB)

6) abstract open book

(Σ, h)

surface w/ boundary

$h: \Sigma \to \Sigma$ diffeomorphism that is identity near boundary

aOB induces OBD:

$$M = \Sigma \times [0,1] \Big/ (x,1) \sim (h(x),0) \quad \cup_g \quad \bigsqcup_{(\#\text{ boundary comp})} (S^1 \times D^2)$$

g gluing map that glues

$(S^1 \times S^1) = \partial(S^1 \times D^2)$ and $p \times S^1 \subset \partial M$

by identifying $S^1 \times \{*\}$ and $p \times \{*\}$

extension of map $\theta: \Sigma \times [0,1] \Big/ (x,1) \sim (h(x),0)$

to $\bigsqcup S^1 \times \{D^2 \setminus 0\}$ gives OBD of M

On the other hand, for given OBD (M, K, θ)

θ ~~restricted to~~ gives us monodromy

map up to isotopy and conjugation by diff.

by fixing local trivializations of θ and „going from one end to the other" in S^1.

-3- (2) Existence of open book decomposition on 3-mfd

First recall

Thm. (Lickorish–Wallace) Every oriented 3-mfd can be obtained from S^3 by ± 1 surgery along some link.

Proof.

We look at genus g Heegaard decompositions

$S^3 = H_1 \cup_f H_2$ $\qquad f: \partial H_1 \xrightarrow{\cong} \partial H_2$

$M = \tilde{H}_1 \cup_g \tilde{H}_2$ $\qquad g: \partial \tilde{H}_1 \xrightarrow{\cong} \partial \tilde{H}_2$

$I: H_1 \to \tilde{H}_1$ diffeomorphism

It is enough to find extension of $I|_{\partial H_2 \cong \partial H_1}$

to $H_2 \setminus \cup V_i \to \tilde{H}_2 \setminus \cup V_i'$

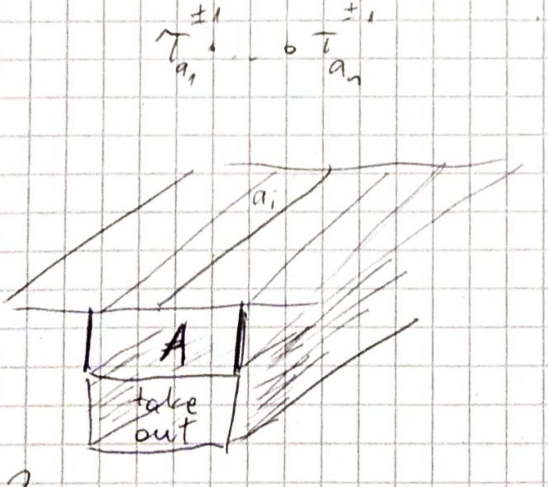
for V_i, V_i' some disjoint solid tori in $H_2, \tilde{H}_2$

i.e. find extension of $g \circ f^{-1}$ to $H_2 \setminus$ solid tori

this is done by „digging" ~~the~~ solid tori beneath ~~certain curves~~ curves $a_1, \dots, a_n$ for which

$g \circ f^{-1}: \Sigma_g \to \Sigma_g$ is isotopic to

$\tau_{a_1}^{\pm 1} \circ \dots \circ \tau_{a_n}^{\pm 1}$ $\quad$ (τ_{a_i} Dehn twist along a_i)



on $A \cong$ nbhd $a_i \times [0,1]$ extend by $\tau_{a_i}^{\pm 1} \times \mathrm{Id}_{[0,1]}$

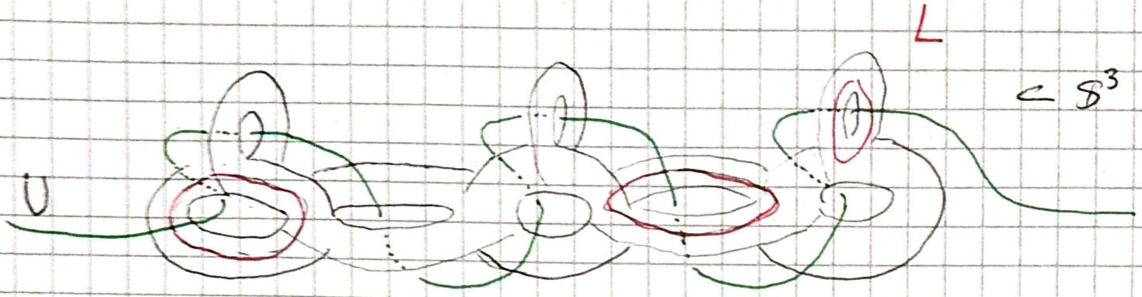
on the rest (B) extend by identity

}

Surgery coeff. is ± 1 because extension is equal to $\tau_{a_i}^{\pm}$ on the upper piece of the tunnel and identity on the other sides

-4- Remark: $MCG(\Sigma_g)$ is generated by $3g-1$ curves

So surgery can be done along link L ~~[struck out]~~ close to the cores of the following body



Moreover, there exists unknot U s.t. link L is braided once around U. Braided means $\exists$ OBD wrt U s.t. link L is transverse to all pages

Thm: Every 3-dim mfd M has OBD over some knot.

proof. $V_1, \dots, V_r \subset S^3$ nbhd of L
$\tilde{V}_1, \dots, \tilde{V}_r \subset M$

$f : S^3 \setminus \bigcup \mathring{V}_i \to M \setminus \bigcup \mathring{\tilde{V}}_i$ diffeomorphism

carries μ meridians of V_i to longitudes of $\tilde{V}_i$ (surgery coef. ± 1)

and let (U, θ) be open book decomp. s.t. L is transverse to pages

transversality $\Rightarrow$ WLOG assume pages intersect V_i along meridians

This induces fibration on

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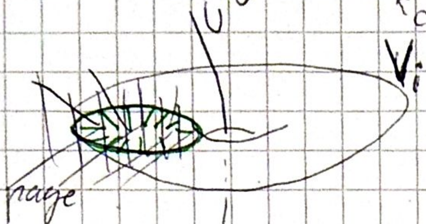
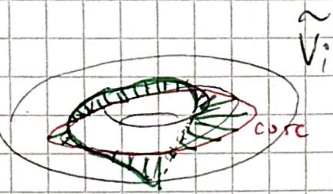
$M \setminus (\tilde{V}_i \cup f(U))$

we will extend it to OBD on M with binding $\bigcup C(\tilde{V}_i) \cup f(U)$ in the following way



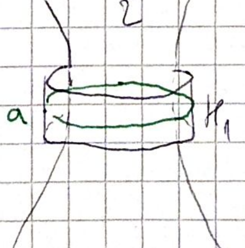
since surgery coeff. is ± 1, image of page intersects $\tilde{V}_i$ at logitude $\mu \pm l$



fibration extends to $V_i \setminus$core and core is in the boundary of each page of M

This proves that there exists OBD of M over some link. To get OBD over knot (page has connected boundary) we use stabilization.

(Σ, h) OBD of M, h monodromy then we can get new OBD by attaching a handle to Σ and taking monodromy $h \circ \tau_a$, where a is a curve on $\Sigma \cup H_1$ that intersects core of the handle once



To see that this induces OBD of the same 3-mfd we look at the following construction.

Murasugi sum

(Σ_1, h_1) and (Σ_2, h_2) abstract open books

a_i arcs in Σ_i with $\partial a_i \subset \partial \Sigma_i$

$R_i \cong [-1,1] \times a_i$ nbhd of a_i

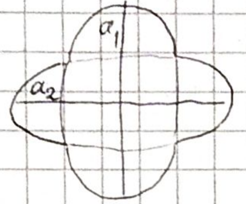
we define a OB

$$\Sigma = \Sigma_1 \cup_g \Sigma_2$$

$$g : R_1 \to R_2$$

$$[-1,1] \times a_1 \to a_2 \times [-1,1]$$

Σ



with monodromy $h = h_2 \circ h_1$

Exercise: It is not hard to show

$$M_{(\Sigma, h)} \cong M_{(\Sigma_1, h_1)} \# M_{(\Sigma_2, h_2)}$$

Murasugi sum of (Σ, h) and a OB obtained from Hopf link OBD gives us stabilization of (Σ, h) (monodromy of Hopf link OBD is τ_a).

therefore $M_{stab} \cong M \# S^3 \cong M$

stabilization doesn't change the mfd.

With stabilization, we can connect disconnected components of $\partial \Sigma$ by gluing handle between them.

$\Rightarrow$ Every 3mfd has OBD over a knot. $\square$

(3.) Contact structures

(M,ω) symplectic mfd. ($d\omega = 0$, ω non-degenerate)

$H: M \to \mathbb{R}$

X_H unique v field s.t.

$\omega(X_H, \ ) = -dH$, φ_t its flow

$\Sigma = H^{-1}(p) \subset M$ hypersurface

$\frac{d}{dt} H(\varphi_t(x)) = -\omega(X_H, X_H) = 0 \Rightarrow X_H$ tangent to Σ

- Question of closed orbits of X_H...

orbits on Σ don't depend on H, but only on ~~topology~~ geometry of Σ, X_H determine unique direction s.t. $\omega(X_H, v) = 0 \ \forall v \in T\Sigma$ (called characteristic line field)

If V v. field on a nbhd U of Σ s.t.

$\mathcal{L}_V \omega = \omega$ and V transverse to Σ

then define $\lambda = i_V \omega$

$d\lambda = d\, i_V \omega = \mathcal{L}_V \omega - i_V d\omega = \omega$ on U

λ satisfies $\lambda \wedge (d\lambda)^{n-1} \neq 0$ on Σ

Such form on $(2n-1)$-dim mfd is called contact form; such hypersurf. contact type hypersurf.

Characteristic line field is determined by

Reeb field R_λ

$$\left(\begin{array}{c}\text{uniquely determined by} \\ d\lambda(R_\lambda, \ ) = 0, \quad \lambda(R_\lambda) \overset{\text{on } \Sigma}{\equiv} 1\end{array}\right)$$

Conjecture: Every contact type hypersurface has closed Reeb orbit.

This point of view on Arnold conjecture ~~[illegible]~~ led to more interest in contact mfds.

The question of existence of contact structure on a mfd is in general hard, we will prove that any 3-mfd has contact structure compatible w/ OBD.

(Y, L, θ) OBD

λ contact form

we say λ is compatible w/ OBD if

$\lambda > 0$ on L and $d\lambda$ is positive area form on every page.

Thm (Thurston - Winkelnkemper, ...) Every OBD on M^3 has compatible contact form.

proof: we construct λ in 2 steps

I step: construct λ on $S := \Sigma \times [0,1] / (x,1) \sim (h(x), 0)$

II step: extend λ to $L \times S^1 \times D^2$

(I) $\mathcal{S} = \{ \eta \in \Omega^1(\Sigma) \mid d\eta$ volume form on Σ, $\eta = t\, d\theta$ ~~near $\partial\Sigma$~~ on $C \}$

fix collar nbhd of $\partial\Sigma$ $C \cong \partial\Sigma \times [0, \frac{1}{2})$ (θ, t)

we will show $\mathcal{S}$ is non-empty and convex

convex ✓ easy

non-empty: Choose volume form Ω on Σ s.t. $\Omega = dt \wedge d\theta$ on C and

$$\int_\Sigma \Omega = \int_{\partial\Sigma} d\theta = \text{length of the boundary}$$

Choose any form η_1 equal to $(1+t)\, d\theta$ on C

$\Omega - d\eta_1$ is compactly supported in $\mathring{\Sigma}$ and

$$\int_\Sigma (\Omega - d\eta_1) = \int_\Sigma \Omega - \int_{\partial\Sigma} \eta_1 = \int_\Sigma \Omega - \int_{\partial\Sigma} d\theta = 0$$

$\Rightarrow \Omega - d\eta_1 = d\eta$ is exact

η compactly supported

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then $\eta_1 + \eta \in S$

for $h: \Sigma \to \Sigma$ that is Id on C

$h^*(\eta_1+\eta) \in S$

convexity $\Rightarrow$ $\tau(\eta_1+\eta) + (1-\tau)\, h^*(\eta_1+\eta) \in S$

Define 1-form on $S = \Sigma \times [0,1] /_\sim$ as

$$\hat{\eta} = \tau(\eta_1+\eta) + (1-\tau)\, h^*(\eta_1+\eta)$$

$\alpha = \hat{\eta} + k\, \pi^* d\tau$ $\pi: S \to S^1$ projection to $\overset{\tau}{[0,1]}/_\sim$

$d\tau$ volume form on S^1

(u, v, w) basis of $T_{(x,\tau)} S$ st.

$$\begin{cases} \pi_*(u) = \pi_*(v) = 0 & (d\hat{\eta}(u,v) > 0) \\ \pi_*(w) \neq 0 & (d\tau(\pi_* w) > 0) \end{cases}$$

$$(\alpha \wedge d\alpha)(u,v,w) = (\hat{\eta} \wedge d\hat{\eta})(u,v,w) +$$
$$+ \; k\, \underbrace{d\tau(\pi_* w)}_{>0}\, \underbrace{d\hat{\eta}(u,v)}_{>0}$$

S^1 compact
$\Rightarrow$ for k large α is contact form on S

and has form $(1+t)\,d\theta + k\,d\tau$ near $\partial\Sigma \times S^1$
(t, θ, τ)

(II) On the glued solid tori we have coordinates

$$S^1 \times (D^2 \setminus 0) \ni (\theta, t, \tau)$$

change coord. slightly $r = 1+t$ $r \in (0, \tfrac{3}{2})$

$\alpha = r\,d\theta + k\,d\tau$

extends to $\overbrace{M \setminus L}^{\text{contact form on}}$ but not to M

We look at 1-forms of the form

$\beta = f_1(r)\, d\theta + f_2(r)\, d\tau$, it's easy to see that

β is contact iff $f_2 f_1' - f_2' f_1 \neq 0$

Around $r=0$ take $f_1 = -1$, $f_2 = r^2$

$$-d\theta + r^2 d\tau = -dz + y\,dx + x\,dy$$

in Cartesian coordinates,

which is contact ✓

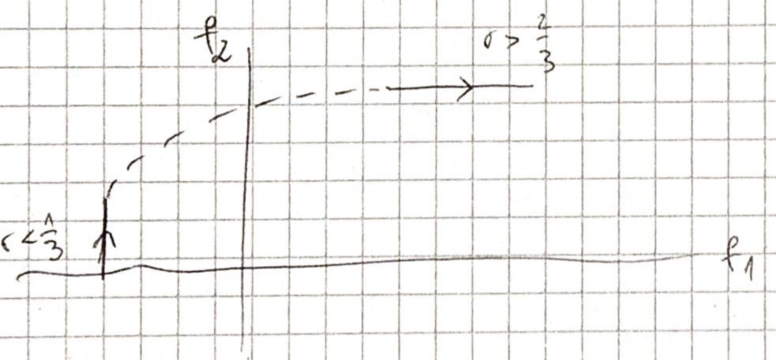
We now extend f_1, f_2

$$f_1(r) = \begin{cases} r & \text{on } \left(\frac{2}{3}, \frac{3}{2}\right) \\ -1 & \text{on } \left[0, \frac{1}{3}\right) \end{cases}$$

$$f_2(r) = \begin{cases} k & \text{on } \left(\frac{2}{3}, \frac{3}{2}\right) \\ r^2 & \text{on } \left[0, \frac{1}{3}\right) \end{cases}$$

to $\left[0, \frac{3}{2}\right)$ s.t. $f_1' f_2 - f_1 f_2' \neq 0$

this is easy



This gives us contact structure on M that is compatible with the given OBD. ◻